

# Acoustic black holes: massless scalar field exact solutions and analogue Hawking radiation

H. S. Vieira<sup>1,2</sup> and V. B. Bezerra<sup>1</sup>

<sup>1</sup> Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, CEP 58051-970, João Pessoa, PB, Brazil

<sup>2</sup> Centro de Ciências, Tecnologia e Saúde, Universidade Estadual da Paraíba, CEP 58233-000, Araruna, PB, Brazil

E-mail: horacio.santana.vieira@hotmail.com and valdir@fisica.ufpb.br

**Abstract.** We obtain the exact solutions of the radial part of the massless Klein-Gordon equation in the spacetime of both three dimensional rotating and four dimensional canonical acoustic black holes, which are given in terms of the confluent Heun functions. From these solutions, we obtain the exact scalar waves near the acoustic horizon. We discuss the analogue Hawking radiation of massless scalar particles and the features of the spectrum associated with the radiation emitted by these acoustic black holes.

PACS numbers: 02.30.Gp, 04.20.Jb, 04.70.-s, 04.80.Cc, 47.35.Rs, 47.90.+a

## 1. Introduction

In a seminal work, Unruh [1] showed that under certain conditions, the equation of motion describing the propagation of sound modes (phonons) on a background hydrodynamic flow, which undergoes a subsonic-supersonic transition, can be written in the same form of the Klein-Gordon equation for a massless scalar field minimally coupled to an effective Lorentzian geometry containing a sonic horizon. This means that this physical system may be considered analogous to astrophysical black holes and as a consequence should be used, in principle, to understand the physics of black holes as well as to realize experiments in laboratory to test some of their properties.

The existence of an event horizon in the hydrodynamic analogue suggests that an interesting phenomenon can be produced which consists in the emission of a thermal flux of phonons, whose temperature is proportional to the gradient of the velocity field at the acoustic horizon, termed analogue Hawking radiation, similar to the radiation emitted by astrophysical black holes, the well know Hawking radiation [2, 3, 4, 5, 6]. Thus, using this analogy, it is possible, in principle, to experimentally verify the analogue Hawking radiation emitted by acoustic black holes, and assuming that the physics which leads to Hawking radiation should be the same of its analogue, we can get some informations about a phenomenon which originates from the combination of quantum mechanics and general relativity, but now at a purely classical level.

It is difficult to observe the Hawking radiation emitted by astrophysical black holes due to the fact that Hawking temperature is seven orders of magnitude smaller than the temperature of the Cosmic Microwave Background (CMB) radiation, for a Schwarzschild black hole with a mass equivalent to the solar mass. Thus, the experimental verification of Hawking radiation emitted by astrophysical black holes could be possible only if these objects have mass much smaller than the solar mass. On the other hand, for acoustic black holes in condensed matter, the analogue Hawking temperature can be of about  $10^{-6}\text{K}$  to  $10^{-7}\text{K}$  (for a review, see [7] and references therein), which is too low to be measured in laboratory, but certainly these values will be measured in a near future. The fact that Hawking radiation from astrophysical black holes has an analogue in hydrodynamic systems has stimulated the realization of several experiments [8, 9] and the suggestion of new experiments involving water waves [10].

Hawking radiation is an effect of kinematical origin which appears in the scope of general relativity connected with the existence of event horizons, and therefore, has not relation with aspects of the dynamic of the Einstein equations. Thus, if we have an analogue system which exhibits an event horizon, we expect the existence of an analogue of Hawking radiation as stated before. Thus, this radiation is a more general phenomenon which occurs whenever an event horizon exists, and therefore, in different analogous gravitational systems. The fact that the temperatures involved in analogue models are very low suggests the realization of experiments involving ultra cold systems, such as Bose-Einstein condensates [11, 12, 13, 14], superfluid helium [15], superconductores [16, 17], polariton superfluid [18] and degenerate Fermi gas

[19]. Otherwise, due to the general character of the Hawking radiation and the real possibility to detect its analogue, many other systems have been investigated [20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

The astrophysical black holes and the corresponding analogue gravity models have many features in common, even possessing different dynamics. As for example, the dynamics of the acoustic black holes are governed by equations of fluid mechanics, while the dynamics of the astrophysical black holes are obtained from Einstein's equations. The fact that these different kinds of black holes possess some fundamental properties in common, associated with the possibility to detect the acoustic black holes in laboratory, which can help us to better understand the physics of a gravitational black hole, motivated a lot of investigation in which concerns the physics of acoustic black holes [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45].

A particular solution [46] of the massless Klein-Gordon equation in the rotating acoustic background was obtained using a tortoise coordinate, in two distinct regions, namely, near the horizon and very far from it, with the proposal to examine the phenomenon of superresonance. In order to investigate the superresonant scattering of acoustic disturbances from a rotating acoustic black hole it was found an approximate solution [47], in the low-frequency limit, which is given in terms of hypergeometric functions. This solution, with appropriate boundary conditions, was also used to compute the quasinormal modes [48] associated with an acoustic rotating black holes.

In our paper we obtain the exact solutions of the Klein-Gordon equation for a massless scalar field for both rotating and canonical acoustic black holes, valid in the whole space, which means between the acoustic event horizon and infinity. Furthermore, these solutions are valid for all frequencies. In this sense, we extend the range in which the solutions are valid as compared with the ones obtained by Basak and Majumdar [47], for the rotating acoustic black hole and additionally the solution of the Klein-Gordon equation in the canonical acoustic black hole background was obtained. They are given in terms of solutions of the Heun equations [49]. Using the radial solution which is given in terms of the confluent Heun functions and taking into account their properties, we study the analogue Hawking radiation of massless scalar particles.

In order to study Hawking radiation, different approaches have been used. As examples, we can mention the following: (i) Hartle-Hawking [50] which used analytic continuation; (ii) Christensen-Fulling which is based on the trace anomaly or conformal anomaly [51]; (iii) Robinson-Wilczek which takes into account the anomaly cancellation [52]; (iv) Tunneling method [53] which uses the semiclassical WKB method, among others. Using these different approaches, Kim and Shin [54] considered the analogue Hawking radiation from the rotating acoustic black hole using the viewpoint of anomaly cancellation method. Ramón *et al.* [55] presented the analogue Hawking radiation derivation from the canonical acoustic black hole using the anomaly and tunneling mechanism approaches. Zhang *et al.* [56] studied Hawking radiation from a rotating acoustic black hole using the analytical continuation method [57].

In this paper, we use just the radial part of the exact solution of the Klein-Gordon

equation, from which we construct the solutions near the acoustic horizons which are appropriate to examine Hawking radiation.

This paper is organized as follows. In the section 2, we introduce the metric that corresponds to a rotating acoustic black hole, and some relevant elements to study the Hawking radiation. We also present the exact solution of the Klein-Gordon equation in this background, taking into account only the radial part. This solution was used to construct the ingoing and outgoing waves, near to the acoustic horizon. We extend the waves solutions from outside to inside of the rotating acoustic black hole, and we derive the dumb body radiation spectra. In the section 3, we do similar calculations in the background of the canonical acoustic black hole, and we derive the corresponding dumb body radiation spectrum. Finally, in section 4, we present our conclusions.

## 2. Rotating acoustic black hole

The solution which describes a (2+1)-dimensional rotating acoustic black hole was obtained by Visser [58] and corresponds to an analogue model associated with a smoothly rotating draining fluid flow with a sink at the origin. Note that this fluid flow is constant in time and cylindrically symmetric, and therefore, represents a vortex line aligned along z-axis, without vorticity and being barotropic and inviscid. The acoustic metric appropriate to represent the effective Lorentzian geometry of a draining bathtub idealized model is given by [58]

$$ds^2 = -c^2 dt^2 + \left(dr - \frac{A}{r}dt\right)^2 + \left(r d\phi - \frac{B}{r}dt\right)^2, \quad (1)$$

where  $c$  is the speed of sound which is constant throughout the fluid flow. The properties of this metric become evident by changing the coordinates, defined by

$$dt \rightarrow dt + \frac{|A|r}{c^2r^2 - A^2}dr, \quad (2)$$

$$d\phi \rightarrow d\phi + \frac{B|A|r}{r(c^2r^2 - A^2)}dr. \quad (3)$$

Using these new coordinates and rescaling time coordinate by  $c$ , the line element that describes a rotating acoustic black hole can be written in the following form

$$ds^2 = -\frac{1}{r^2} \left(\Delta - \frac{B^2}{r^2}\right) dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\phi^2 - 2\frac{B}{c} d\phi dt, \quad (4)$$

where

$$\Delta = r^2 - \frac{A^2}{c^2}. \quad (5)$$

Taking  $g_{00}$  in Eq. (4) equal to zero, we obtain the radius of the ergosphere, which is given by

$$r_e = \frac{\sqrt{A^2 + B^2}}{c}. \quad (6)$$

It is worth calling attention to the fact that the sign of  $A$  is irrelevant in defining the ergosphere and ergo-region, since does not matter if the vortex core is a source or a sink. The metric has a (coordinate) singularity, that is, the acoustic event horizon forms once the radial component of the fluid velocity exceeds the speed of sound, which occurs at

$$r_h = \frac{|A|}{c} . \quad (7)$$

Here, the sign of  $A$  must be taken into account. For  $A < 0$  we are dealing with a future acoustic horizon, that is, an acoustic black hole; while for  $A > 0$  we are dealing with a past event horizon, that is, an acoustic white hole.

Hence, from Eq. (5), we have that the acoustic horizon surface of the rotating acoustic black hole is obtained from the condition

$$\Delta = (r - r_+)(r - r_-) = 0 , \quad (8)$$

whose solutions are

$$r_+ = r_h , \quad (9a)$$

$$r_- = -r_h , \quad (9b)$$

where  $r_h$  is given by Eq. (7), and  $r_{\pm}$  correspond to the acoustic events or Cauchy horizons of the rotating acoustic black hole.

The gravitational acceleration,  $\kappa_h$ , on the acoustic black hole horizon surface,  $r_+ = r_h$ , is given by

$$\kappa_h \equiv \left. \frac{1}{2} \frac{1}{r_h^2} \frac{d\Delta}{dr} \right|_{r=r_h} = \frac{1}{r_h} = \frac{c}{|A|} . \quad (10)$$

Then, an acoustic event horizon will emit an analogue Hawking radiation corresponding to a thermal bath of phonons at a temperature

$$T_h = \frac{\kappa_h}{2\pi} . \quad (11)$$

### 2.1. Exact solutions of the Klein-Gordon equation

In what follows we will consider the covariant Klein-Gordon equation, that describes the behavior of scalar fields in a curved spacetime, which has the form

$$\left[ \frac{1}{\sqrt{-g}} \partial_\rho (g^{\rho\sigma} \sqrt{-g} \partial_\sigma) \right] \Psi = 0 . \quad (12)$$

Thus, the covariant Klein-Gordon equation in the spacetime of a rotating acoustic black hole given by the line element (4), can be written as

$$\left[ -\frac{r^3}{\Delta} \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial r} \left( \frac{\Delta}{r} \frac{\partial}{\partial r} \right) + \left( \frac{1}{r} - \frac{B^2}{c^2 r \Delta} \right) \frac{\partial^2}{\partial \phi^2} - \frac{2Br}{c\Delta} \frac{\partial^2}{\partial \phi \partial t} \right] \Psi(\mathbf{r}, t) = 0 . \quad (13)$$

The spacetime under consideration is stationary, so the time dependence that solves Eq. (13) may be separated as  $e^{-i\omega t}$ , where  $\omega$  is the energy of the particles in the units

chosen and we are assuming that  $\omega > 0$ . Moreover, its axisymmetry permits us to separate the solution in  $\phi$  as  $e^{im\phi}$ , where  $m$  is a real constant that is not restricted to assume only a discrete set of values, because we are working with only two space dimensions. Thus, we can make the following ansatz

$$\Psi(\mathbf{r}, t) = R(r)e^{im\phi}e^{-i\omega t} . \quad (14)$$

Substituting Eq. (14) into (13), we find that

$$\frac{d}{dr} \left( \frac{\Delta}{r} \frac{dR}{dr} \right) + \left( \frac{\omega^2 r^3}{\Delta} - \frac{m^2}{r} + \frac{m^2 B^2}{c^2 r \Delta} - \frac{2mB\omega r}{c\Delta} \right) R = 0 . \quad (15)$$

Now, let us obtain the exact and general solution for the radial part of Klein-Gordon equation given by Eq. (15). To do this, we will not take asymptotic limits, nor assume any condition on the frequency. On the contrary, we will consider the whole space, as well as the whole range of frequencies (or energies in the natural units), that is,  $0 < \omega \leq \infty$ , and thus the solutions for low frequency regime considered by Lepe and Saavedra [48] can be obtained as a particular case.

Equation (15) has an undesirable singularity at the origin, which hinders considerably its transformation to a Heun-type equation. However, introducing a new radial coordinate,  $x$ , such that

$$x = \frac{r^2}{2} , \quad (16)$$

and using this new coordinate, Eq. (5) turns into

$$\Delta = 2x - \frac{A^2}{c^2} . \quad (17)$$

Thus, from Eq. (17), we have that the new horizon surface of the rotating acoustic black hole is obtained from the condition

$$\Delta = 2(x - x_h) = 0 , \quad (18)$$

where

$$x_h = \frac{1}{2} r_h^2 \quad (19)$$

is the root of  $\Delta$  and corresponds to the new acoustic event horizon of the rotating acoustic black hole. Hence, using Eq. (18), we can write down Eq. (15) as

$$\begin{aligned} & \frac{d^2 R}{dx^2} + \left( \frac{1}{x - x_h} \right) \frac{dR}{dx} \\ & + \left[ \frac{m^2 (B^2 + 2c^2 x_h)}{8c^2 x_h^2} \frac{1}{x} + \frac{-B^2 m^2 - 2c^2 m^2 x_h + 4c^2 x_h^2 \omega^2}{8c^2 x_h^2} \frac{1}{x - x_h} \right. \\ & \left. + \frac{B^2 m^2 - 4Bcmx_h \omega + 4c^2 x_h^2 \omega^2}{8c^2 x_h} \frac{1}{(x - x_h)^2} \right] R = 0 . \end{aligned} \quad (20)$$

This equation has singularities at  $x = (x_1, x_2) = (x_h, 0)$ . The transformation of Eq. (20) to a Heun-type equation is achieved by setting the following homographic substitution

$$z = \frac{x - a_1}{a_2 - a_1} = \frac{x - x_h}{0 - x_h} . \quad (21)$$

Thus, we can write Eq. (20) as

$$\begin{aligned} & \frac{d^2 R}{dz^2} + \frac{1}{z} \frac{dR}{dz} \\ & + \left\{ \frac{B^2 m^2 + 2c^2 m^2 x_h - 4c^2 x_h^2 \omega^2}{8c^2 x_h} \frac{1}{z} + \frac{-m^2 (B^2 + 2c^2 x_h)}{8c^2 x_h} \frac{1}{z-1} \right. \\ & \left. - \left[ i \left( \frac{2cx_h \omega - mB}{2c\sqrt{2x_h}} \right) \right]^2 \frac{1}{z^2} \right\} R = 0 . \end{aligned} \quad (22)$$

In what follows, let us perform an appropriate transformation in order to reduce the power of the term proportional to  $1/z^2$ . This transformation is the *s-homotopic transformation* of the dependent variable  $R(z) \mapsto U(z)$  such that

$$R(z) = z^{A_1} U(z) , \quad (23)$$

where the coefficient  $A_1$  is given by

$$A_1 = i \left( \frac{2cx_h \omega - mB}{2c\sqrt{2x_h}} \right) . \quad (24)$$

In this case, the function  $U(z)$  satisfies the following equation

$$\begin{aligned} & \frac{d^2 U}{dz^2} + \left( \frac{2A_1 + 1}{z} \right) \frac{dU}{dz} \\ & + \left[ \frac{B^2 m^2 + 2c^2 m^2 x_h - 4c^2 x_h^2 \omega^2}{8c^2 x_h} \frac{1}{z} - \frac{m^2 (B^2 + 2c^2 x_h)}{8c^2 x_h} \frac{1}{z-1} \right] U = 0 , \end{aligned} \quad (25)$$

which is similar to the confluent Heun equation [59]

$$\frac{d^2 U}{dz^2} + \left( \alpha + \frac{\beta + 1}{z} + \frac{\gamma + 1}{z-1} \right) \frac{dU}{dz} + \left( \frac{\mu}{z} + \frac{\nu}{z-1} \right) U = 0 , \quad (26)$$

where  $U(z) = \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z)$  are the confluent Heun functions, with the parameters  $\alpha, \beta, \gamma, \delta$  and  $\eta$ , related to  $\mu$  and  $\nu$  by

$$\mu = \frac{1}{2}(\alpha - \beta - \gamma + \alpha\beta - \beta\gamma) - \eta , \quad (27a)$$

$$\nu = \frac{1}{2}(\alpha + \beta + \gamma + \alpha\gamma + \beta\gamma) + \delta + \eta , \quad (27b)$$

according to the standard package of the **Maple**<sup>TM</sup>17.

Thus, the general solution of the radial part of the Klein-Gordon equation for a massless scalar field in the spacetime of a rotating acoustic black hole, in the region exterior to the acoustic event horizon, given by Eq. (22), over the entire range  $0 \leq z < \infty$ , can be written as

$$\begin{aligned} R(z) &= z^{\frac{1}{2}\beta} \\ &\times \{ C_1 \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z) + C_2 z^{-\beta} \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta; z) \} , \end{aligned} \quad (28)$$

where  $C_1$  and  $C_2$  are constants, and the parameters  $\alpha, \beta, \gamma, \delta$ , and  $\eta$  are now given by:

$$\alpha = 0 ; \quad (29a)$$

$$\beta = i \left( \frac{2cx_h\omega - mB}{c\sqrt{2x_h}} \right) ; \quad (29b)$$

$$\gamma = -1 ; \quad (29c)$$

$$\delta = -\frac{x_h\omega^2}{2} ; \quad (29d)$$

$$\eta = \frac{1}{8} \left[ m^2 \left( -\frac{B^2}{c^2 x_h} - 2 \right) + 4x_h\omega^2 + 4 \right] . \quad (29e)$$

These two functions form linearly independent solutions of the confluent Heun differential equation due to the fact that  $\beta$  is not necessarily an integer. If we consider the expansion in power series of the confluent Heun functions with respect to the independent variable  $z$ , in a neighborhood of the regular singular point  $z = 0$  [60], we can write

$$\begin{aligned} \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z) = & 1 + \frac{1}{2} \frac{(-\alpha\beta + \beta\gamma + 2\eta - \alpha + \beta + \gamma)}{(\beta + 1)} z \\ & + \frac{1}{8} \frac{1}{(\beta + 1)(\beta + 2)} (\alpha^2\beta^2 - 2\alpha\beta^2\gamma + \beta^2\gamma^2 \\ & - 4\eta\alpha\beta + 4\eta\beta\gamma + 4\alpha^2\beta - 2\alpha\beta^2 - 6\alpha\beta\gamma \\ & + 4\beta^2\gamma + 4\beta\gamma^2 + 4\eta^2 - 8\eta\alpha + 8\eta\beta + 8\eta\gamma \\ & + 3\alpha^2 - 4\alpha\beta - 4\alpha\gamma + 3\beta^2 + 4\beta\delta \\ & + 10\beta\gamma + 3\gamma^2 + 8\eta + 4\beta + 4\delta + 4\gamma) z^2 + \dots , \quad (30) \end{aligned}$$

which is a useful form to be used in the discussion of the analogue Hawking radiation.

## 2.2. Analogue Hawking radiation

We will consider the massless scalar field near the acoustic horizon in order to discuss the analogue Hawking radiation. From Eqs. (21) and (30), we can see that the radial solution given by Eq. (28), near the acoustic event horizon, that is, when  $r \rightarrow r_h \Rightarrow x \rightarrow x_h \Rightarrow z \rightarrow 0$ , behaves asymptotically as

$$R(r) \sim C_1 (r - r_h)^{\beta/2} + C_2 (r - r_h)^{-\beta/2} , \quad (31)$$

where we are considering contributions only of the first term in the expansion, and all constants are included in  $C_1$  and  $C_2$ . Thus, considering the solution of the time dependence, near the rotating acoustic black hole event horizon  $r_h$ , we can write

$$\Psi = e^{-i\omega t} (r - r_h)^{\pm\beta/2} . \quad (32)$$

From Eq. (29b), for the parameter  $\beta$ , we obtain

$$\beta = ir_h \left( \omega - m \frac{B}{cr_h^2} \right) . \quad (33)$$

Then, substituting Eqs. (7) and (10) into (33), we get

$$\beta = \frac{i}{\kappa_h} (\omega - \omega_h) , \quad (34)$$



where

$$\omega_h = m\Omega_h , \quad (35)$$

with

$$\Omega_h = \frac{Bc}{A^2} \quad (36)$$

being the dragging angular velocity of the acoustic event horizon.

Therefore, on the rotating acoustic black hole horizon surface, the ingoing and outgoing wave solutions are

$$\Psi_{in} = e^{-i\omega t} (r - r_h)^{-\frac{i}{2\kappa_h}(\omega - \omega_h)} , \quad (37)$$

$$\Psi_{out}(r > r_h) = e^{-i\omega t} (r - r_h)^{\frac{i}{2\kappa_h}(\omega - \omega_h)} . \quad (38)$$

These solutions for the scalar fields near the acoustic horizon will be useful to investigate Hawking radiation of massless scalar particles. It is worth calling attention to the fact that we are using the analytical solution of the radial part of Klein-Gordon equation in the spacetime under consideration, differently from the calculations usually done in the literature [46, 48].

Using the definitions of the tortoise and Eddington-Finkelstein coordinates, given by [54]

$$dr_* = \frac{r^2}{\Delta} dr , \quad (39)$$

we have

$$\ln(r - r_h) = \frac{1}{r_h^2} \frac{d\Delta}{dr} \Big|_{r=r_h} r_* = 2\kappa_h r_* , \quad (40)$$

$$\hat{r} = \frac{\omega - \omega_h}{\omega} r_* , \quad (41)$$

$$v = t + \hat{r} , \quad (42)$$

and thus the following ingoing wave solution can be obtained

$$\begin{aligned} \Psi_{in} &= e^{-i\omega v} e^{i\omega \hat{r}} (r - r_h)^{-\frac{i}{2\kappa_h}(\omega - \omega_h)} \\ &= e^{-i\omega v} e^{i(\omega - \omega_h)r_*} (r - r_h)^{-\frac{i}{2\kappa_h}(\omega - \omega_h)} \\ &= e^{-i\omega v} (r - r_h)^{\frac{i}{2\kappa_h}(\omega - \omega_h)} (r - r_h)^{-\frac{i}{2\kappa_h}(\omega - \omega_h)} \\ &= e^{-i\omega v} . \end{aligned} \quad (43)$$

The outgoing wave solution is given by

$$\begin{aligned} \Psi_{out}(r > r_h) &= e^{-i\omega v} e^{i\omega \hat{r}} (r - r_h)^{\frac{i}{2\kappa_h}(\omega - \omega_h)} \\ &= e^{-i\omega v} e^{i(\omega - \omega_h)r_*} (r - r_h)^{\frac{i}{2\kappa_h}(\omega - \omega_h)} \\ &= e^{-i\omega v} (r - r_h)^{\frac{i}{2\kappa_h}(\omega - \omega_h)} (r - r_h)^{\frac{i}{2\kappa_h}(\omega - \omega_h)} \\ &= e^{-i\omega v} (r - r_h)^{\frac{i}{\kappa_h}(\omega - \omega_h)} . \end{aligned} \quad (44)$$

Note that, if we put  $Q = 0$  into Eqs. (96) and (97) of Ref. [61], the resulting solutions are exactly analogous to solutions above given by Eqs. (43) and (44).

### 2.3. Analytic extension and radiation spectrum

Now, let us obtain by analytic continuation a real damped part of the outgoing wave solution of the massless scalar field which will be used to construct an explicit expression for the decay rate  $\Gamma_h$ . This real damped part corresponds (at least in part) to the temporal contribution to the decay rate [56] found by the tunneling method used to investigate the analogue Hawking radiation.

From Eq. (44), we see that this solution is not analytical in the acoustic event horizon  $r = r_h$ . By analytic continuation, rotating by an angle  $-\pi$  through the lower-half complex  $r$  plane, we obtain

$$(r - r_h) \rightarrow |r - r_h| e^{-i\pi} = (r_h - r) e^{-i\pi} . \quad (45)$$

Thus, the outgoing wave solution on the acoustic horizon surface  $r_h$  is

$$\Psi_{out}(r < r_h) = e^{-i\omega v} (r_h - r)^{\frac{i}{\kappa_h}(\omega - \omega_h)} e^{\frac{\pi}{\kappa_h}(\omega - \omega_h)} . \quad (46)$$

Equations (44) and (46) describe the outgoing wave outside and inside of the rotating acoustic black hole event horizon, respectively. Therefore, for an outgoing wave of a particle with energy  $\omega > 0$ , the outgoing decay rate or the relative scattering probability of the scalar wave at the acoustic event horizon surface,  $r = r_h$ , is given by

$$\Gamma_h = \left| \frac{\Psi_{out}(r > r_h)}{\Psi_{out}(r < r_h)} \right|^2 = e^{-\frac{2\pi}{\kappa_h}(\omega - \omega_h)} , \quad (47)$$

which is result already formally obtained in the literature [56] in different context, and is analogous to the one obtained in [61] for an astrophysical black hole.

According to the Damour-Ruffini-Sannan method [62, 63] for astrophysical black holes, a correct wave describing a particle flying off of the rotating acoustic black hole showed be given by

$$\begin{aligned} \Psi_\omega(r) = N_\omega [ & H(r - r_h) \Psi_\omega^{out}(r - r_h) \\ & + H(r_h - r) \Psi_\omega^{out}(r_h - r) e^{\frac{\pi}{\kappa_h}(\omega - \omega_h)} ] , \end{aligned} \quad (48)$$

where  $N_\omega$  is the normalization constant, such that

$$\langle \Psi_{\omega_1}(r) | \Psi_{\omega_2}(r) \rangle = -\delta(\omega_1 - \omega_2) , \quad (49)$$

with  $H(x)$  being the Heaviside function and  $\Psi_\omega^{out}(x)$  the normalized wave functions given, from Eq. (44), by

$$\Psi_\omega^{out}(x) = e^{-i\omega v} x^{\frac{i}{\kappa_h}(\omega - \omega_h)} . \quad (50)$$

Thus, from the normalization condition

$$\langle \Psi_\omega(r) | \Psi_\omega(r) \rangle = 1 = |N_\omega|^2 \left[ e^{\frac{2\pi}{\kappa_h}(\omega - \omega_h)} - 1 \right] , \quad (51)$$

we get the resulting analogue Hawking radiation spectrum of scalar particles, which is given by

$$|N_\omega|^2 = \frac{1}{e^{\frac{2\pi}{\kappa_h}(\omega - \omega_h)} - 1} = \frac{1}{e^{\frac{\hbar(\omega - \omega_h)}{k_B T_h}} - 1} , \quad (52)$$

where the Boltzmann's and Planck's constants were reintroduced.

Therefore, we can see that the resulting analogue Hawking radiation spectrum of massless scalar particles, the phonons, has a thermal character, analogous to the black body spectrum, where  $k_B T_h = \hbar \kappa_h / 2\pi$ .

### 3. Canonical acoustic black hole

A solution for a spherically symmetric flow of incompressible fluid, called canonical acoustic black hole, was found by Visser in [58]. The acoustic metric which appropriately describes this situation is given by

$$ds^2 = -c^2 dt^2 + \left( dr \pm c \frac{r_0^2}{r^2} dt \right)^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (53)$$

where  $c$  is the speed of sound and is constant throughout the fluid flow, and is defined in terms of the velocity  $v$  as

$$v = c \frac{r_0^2}{r^2} . \quad (54)$$

Using the Schwarzschild time coordinate  $\tau$  instead of the laboratory time  $t$ , and doing the coordinates transformation [58]

$$d\tau = dt \pm \frac{r_0^2/r^2}{c[1 - (r_0^4/r^4)]} dr , \quad (55)$$

the line element that describes a canonical acoustic black hole can be rewritten as

$$ds^2 = -\frac{c^2}{r^4} \Delta d\tau^2 + \frac{r^4}{\Delta} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (56)$$

where

$$\Delta = r^4 - r_0^4 . \quad (57)$$

Obviously, the form of this metric is different from that of the standard geometries typically considered in general relativity.

From Eq. (56) we conclude that the radius of the acoustic event horizon is given by

$$r_h = r_0 , \quad (58)$$

where  $r_0$  is obtained from Eq. (54), and corresponds to the event horizon of the canonical acoustic black hole.

The gravitational acceleration,  $\kappa_0$ , on the acoustic black hole horizon surface,  $r_0$ , is given by [64]

$$\kappa_0 \equiv \left| \frac{\partial v^r}{\partial r} \right|_{r=r_0} = \frac{2c}{r_0} . \quad (59)$$

Then, an acoustic event horizon will emit analogue Hawking radiation in the form of a thermal bath of phonons at a temperature

$$T_0 = \frac{\kappa_0}{2\pi c} = \frac{1}{\pi r_0} . \quad (60)$$

### 3.1. Exact solutions of the Klein-Gordon equation

The covariant Klein-Gordon equation in the spacetime of a canonical acoustic black hole given by the line element (56), can be written as

$$\left[ -\frac{r^6}{c^2 \Delta} \frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial r} \left( \frac{\Delta}{r^2} \frac{\partial}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \Psi(\mathbf{r}, \tau) = 0 . \quad (61)$$

The spacetime under consideration is static, so the time dependence that solves Eq. (61) may be separated as  $e^{-i\omega\tau}$ , where  $\omega$  is the energy of the particles in the units chosen. Moreover, rotational invariance with respect to  $\phi$  implies that the solution in  $\phi$  is  $e^{im\phi}$ , where  $m = \pm 1, \pm 2, \pm 3, \dots$ , is the azimuthal quantum number. Thus, the general angular solution is given in terms of the spherical harmonic function  $Y_m^l(\theta, \phi) = P_m^l(\cos \theta) e^{im\phi}$ , where  $l$  is an integer such that  $|m| \leq l$  [65]. Therefore,  $\Psi(\mathbf{r}, \tau)$  can be written as

$$\Psi(\mathbf{r}, \tau) = R(r) Y_m^l(\theta, \phi) e^{-i\omega\tau} . \quad (62)$$

Substituting Eq. (62) into (61), we find that

$$\frac{d}{dr} \left( \frac{\Delta}{r^2} \frac{dR}{dr} \right) + \left( \frac{\omega^2 r^6}{c^2 \Delta} - \lambda_{lm} \right) R = 0 , \quad (63)$$

where  $\lambda_{lm} = l(l+1)$ .

Now, let us obtain the exact and general solution for the radial part of Klein-Gordon equation given by Eq. (63). This equation has four singularities, which hinders considerably its transformation to a Heun-type equation. Then, we define a new radial coordinate,  $x$ , such that

$$x = \frac{r^2}{2} . \quad (64)$$

Using this new coordinate, Eq. (57) can be rewritten as

$$\Delta = 4(x^2 - x_0) , \quad (65)$$

where

$$x_0 = \frac{1}{4} r_0^4 . \quad (66)$$

Thus, from Eq. (65), we have that the new acoustic horizon surface equation of the canonical acoustic black hole is obtained from the condition

$$\Delta = 4(x - x_+)(x - x_-) = 0 , \quad (67)$$

whose solutions are

$$x_+ = x_0 , \quad (68a)$$

$$x_- = -x_0 , \quad (68b)$$

and correspond to the new event horizons of the canonical acoustic black hole. Hence, using Eq. (67), we can write down Eq. (63) as

$$\begin{aligned} & \frac{d^2 R}{dx^2} + \left( \frac{1}{x - x_+} + \frac{1}{x - x_-} \right) \frac{dR}{dx} \\ & + \frac{1}{(x - x_+)(x - x_-)} \left[ \frac{\omega^2}{2c^2} x - \frac{c^2 \lambda_{lm} - 2x_+ \omega^2 - 2x_- \omega^2}{4c^2} \right. \\ & \left. + \frac{x_+^3 \omega^2}{2c^2(x_+ - x_-)} \frac{1}{x - x_+} - \frac{x_-^3 \omega^2}{2c^2(x_+ - x_-)} \frac{1}{x - x_-} \right] R = 0 . \end{aligned} \quad (69)$$

This equation has singularities at  $x = (a_1, a_2) = (x_+, x_-)$ , and at  $x = \infty$ . The transformation of Eq. (69) to a Heun-type equation is achieved by setting the following homographic substitution

$$z = \frac{x - a_1}{a_2 - a_1} = \frac{x - x_+}{x_- - x_+} . \quad (70)$$

Thus, we can write Eq. (69) as

$$\begin{aligned} & \frac{d^2 R}{dz^2} + \left( \frac{1}{z} + \frac{1}{z - 1} \right) \frac{dR}{dz} \\ & + \left\{ \frac{c^2 \lambda_{lm} x_+^2 - 2c^2 \lambda_{lm} x_+ x_- + c^2 \lambda_{lm} x_-^2 - 2x_+^3 \omega^2 + 6x_+^2 x_- \omega^2}{4c^2(x_+ - x_-)^2} \frac{1}{z} \right. \\ & + \frac{-c^2 \lambda_{lm} x_+^2 + 2c^2 \lambda_{lm} x_+ x_- - c^2 \lambda_{lm} x_-^2 - 6x_+ x_-^2 \omega^2 + 2x_-^3 \omega^2}{4c^2(x_+ - x_-)^2} \frac{1}{z - 1} \\ & \left. - \left[ i \frac{x_+^{3/2} \omega}{\sqrt{2}c(x_+ - x_-)} \right]^2 \frac{1}{z^2} + \left[ i \frac{x_-^{3/2} \omega}{\sqrt{2}c(x_+ - x_-)} \right]^2 \frac{1}{(z - 1)^2} \right\} R = 0 . \end{aligned} \quad (71)$$

Now, let us perform a transformation in order to reduce the power of the terms proportional to  $1/z^2$  and  $1/(z - 1)^2$ . This transformation is the *s-homotopic transformation* of the dependent variable  $R(z) \mapsto U(z)$  such that

$$R(z) = z^{A_1} (z - 1)^{A_2} U(z) , \quad (72)$$

where the coefficients  $A_1$  and  $A_2$  are given by

$$A_1 = i \frac{x_+^{3/2} \omega}{\sqrt{2}c(x_+ - x_-)} , \quad (73)$$

$$A_2 = i \frac{x_-^{3/2} \omega}{\sqrt{2}c(x_+ - x_-)} . \quad (74)$$

In this case, the function  $U(z)$  satisfies the following equation

$$\begin{aligned} & \frac{d^2 U}{dz^2} + \left( \frac{2A_1 + 1}{z} + \frac{2A_2 + 1}{z - 1} \right) \frac{dU}{dz} \\ & + \left[ \frac{-A_1 - A_2 - 2A_1 A_2 + A_3}{z} + \frac{A_1 + A_2 + 2A_1 A_2 + A_4}{z - 1} \right] U = 0 , \end{aligned} \quad (75)$$

where the coefficients  $A_3$  and  $A_4$  are given by

$$A_3 = \frac{c^2 \lambda_{lm} x_+^2 - 2c^2 \lambda_{lm} x_+ x_- + c^2 \lambda_{lm} x_-^2 - 2x_+^3 \omega^2 + 6x_+^2 x_- \omega^2}{4c^2(x_+ - x_-)^2} , \quad (76)$$

$$A_4 = \frac{-c^2 \lambda_{lm} x_+^2 + 2c^2 \lambda_{lm} x_+ x_- - c^2 \lambda_{lm} x_-^2 - 6x_+ x_-^2 \omega^2 + 2x_-^3 \omega^2}{4c^2 (x_+ - x_-)^2} . \quad (77)$$

Note that Eq. (75) is similar to the confluent Heun equation (26). Thus, the general solution of the radial part of the Klein-Gordon equation for a massless scalar particle in the spacetime of a canonical acoustic black hole, in the exterior region of the acoustic event horizon, given by Eq. (71) over the entire range  $0 \leq z < \infty$ , can be written as

$$R(z) = z^{\frac{1}{2}\beta} (z-1)^{\frac{1}{2}\gamma} \times \{C_1 \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z) + C_2 z^{-\beta} \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta; z)\} , \quad (78)$$

where  $C_1$  and  $C_2$  are constants, and the parameters  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , and  $\eta$  are now given by:

$$\alpha = 0 ; \quad (79a)$$

$$\beta = i\sqrt{2} \frac{x_+^{3/2} \omega}{c(x_+ - x_-)} ; \quad (79b)$$

$$\gamma = i\sqrt{2} \frac{x_-^{3/2} \omega}{c(x_+ - x_-)} ; \quad (79c)$$

$$\delta = -\frac{\omega^2 (x_+ - x_-)}{2c^2} ; \quad (79d)$$

$$\eta = \frac{2x_+^2 \omega^2 (x_+ - 3x_-) - c^2 \lambda (x_+ - x_-)^2}{4c^2 (x_+ - x_-)^2} . \quad (79e)$$

These two functions form linearly independent solutions of the confluent Heun differential equation provided  $\beta$  is not integer, which is a condition satisfied by this parameter because there is no physical reason to impose that  $\beta$  should be an integer.

### 3.2. Analogue Hawking radiation

Now, let us consider the massless scalar field near the horizon in order to discuss the analogue Hawking radiation. To do this, from Eqs. (70) and (30) we can see that the radial solution given by Eq. (78), near the acoustic event horizon, that is, when  $r \rightarrow r_0 \Rightarrow x \rightarrow x_+ \Rightarrow z \rightarrow 0$ , behaves asymptotically as

$$R(r) \sim C_1 (r - r_0)^{\beta/2} + C_2 (r - r_0)^{-\beta/2} , \quad (80)$$

where we are considering contributions arising only from the first term in the expansion, and all constants are included in  $C_1$  and  $C_2$ . Thus, taking into account the time dependence of the solution, near the canonical acoustic black hole event horizon  $r_0$ , we have

$$\Psi = e^{-i\omega\tau} (r - r_0)^{\pm\beta/2} . \quad (81)$$

From Eq. (79b), for the parameter  $\beta$ , we obtain

$$\beta = i \frac{r_0}{2c} \omega . \quad (82)$$

Then, substituting Eqs. (58) and (59) into (82), we get

$$\beta = \frac{i}{\kappa_0} \omega . \quad (83)$$

Therefore, on the canonical acoustic black hole horizon surface, the ingoing and outgoing wave solutions are

$$\Psi_{in} = e^{-i\omega\tau} (r - r_0)^{-\frac{i}{2\kappa_0}\omega} , \quad (84)$$

$$\Psi_{out}(r > r_0) = e^{-i\omega\tau} (r - r_0)^{\frac{i}{2\kappa_0}\omega} . \quad (85)$$

These solutions for the scalar fields near the acoustic horizon obtained from the analytical solution of the radial part of the Klein-Gordon equation in the background under consideration will be useful to investigate analogue Hawking radiation.

Using the definitions of the tortoise and Eddington-Finkelstein coordinates, given by

$$dr_* = \frac{r^4}{c} \frac{1}{\Delta} dr , \quad (86)$$

we have

$$\ln(r - r_0) = \frac{c}{r_0^4} \frac{d\Delta}{dr} \Big|_{r=r_0} r_* = 2\kappa_0 r_* , \quad (87)$$

$$\hat{r} = \frac{\omega - \omega_0}{\omega} r_* , \quad (88)$$

$$v = \tau + \hat{r} , \quad (89)$$

and thus the following ingoing wave solution can be obtained

$$\begin{aligned} \Psi_{in} &= e^{-i\omega v} e^{i\omega\hat{r}} (r - r_0)^{-\frac{i}{2\kappa_0}\omega} \\ &= e^{-i\omega v} e^{i\omega r_*} (r - r_0)^{-\frac{i}{2\kappa_0}\omega} \\ &= e^{-i\omega v} (r - r_0)^{\frac{i}{2\kappa_0}\omega} (r - r_0)^{-\frac{i}{2\kappa_0}\omega} \\ &= e^{-i\omega v} . \end{aligned} \quad (90)$$

The outgoing wave solution is given by

$$\begin{aligned} \Psi_{out}(r > r_0) &= e^{-i\omega v} e^{i\omega\hat{r}} (r - r_0)^{\frac{i}{2\kappa_0}\omega} \\ &= e^{-i\omega v} e^{i\omega r_*} (r - r_0)^{\frac{i}{2\kappa_0}\omega} \\ &= e^{-i\omega v} (r - r_0)^{\frac{i}{2\kappa_0}\omega} (r - r_0)^{\frac{i}{2\kappa_0}\omega} \\ &= e^{-i\omega v} (r - r_0)^{\frac{i}{\kappa_0}\omega} . \end{aligned} \quad (91)$$

Note that, if we put  $a = Q = 0$  into Eqs. (96) and (97) of Ref. [61], the resulting solutions are exactly analogous to solutions given by Eqs. (90) and (91).

### 3.3. Analytic extension and radiation spectrum

Now, we obtain by analytic continuation a real damped part of the outgoing wave solution of the massless scalar field which will be used to construct an explicit expression for the decay rate  $\Gamma_0$ . This real damped part corresponds (at least in part) to the temporal contribution to the decay rate [55] found by the tunneling method used to investigate the analogue Hawking radiation.

From Eq. (91), we see that this solution is not analytical in the acoustic event horizon,  $r = r_0$ . By analytic continuation, rotation by an angle  $-\pi$  through the lower-half complex  $r$  plane, give us

$$(r - r_0) \rightarrow |r - r_0| e^{-i\pi} = (r_0 - r) e^{-i\pi} . \quad (92)$$

Thus, the outgoing wave solution on the acoustic horizon surface  $r_0$  is

$$\Psi_{out}(r < r_0) = e^{-i\omega v} (r_0 - r)^{\frac{i}{\kappa_0}\omega} e^{\frac{\pi}{\kappa_0}\omega} . \quad (93)$$

Equations (91) and (93) describe the outgoing wave outside and inside of the canonical acoustic black hole, respectively. Therefore, for an outgoing wave of a particle with energy  $\omega > 0$ , the outgoing decay rate or the relative scattering probability of the scalar wave at the acoustic event horizon surface  $r = r_0$ , is given by

$$\Gamma_0 = \left| \frac{\Psi_{out}(r > r_0)}{\Psi_{out}(r < r_0)} \right|^2 = e^{-\frac{2\pi}{\kappa_0}\omega} , \quad (94)$$

which is a result already formally obtained in the literature [55] in different context, and is analogous to the obtained in [61] for an astrophysical black hole.

According to the Damour-Ruffini-Sannan method [62, 63] for astrophysical black holes, a correct wave describing a particle flying off of the canonical acoustic black hole is given by

$$\begin{aligned} \Psi_\omega(r) = N_\omega [ & H(r - r_0) \Psi_\omega^{out}(r - r_0) \\ & + H(r_0 - r) \Psi_\omega^{out}(r_0 - r) e^{\frac{\pi}{\kappa_0}\omega} ] , \end{aligned} \quad (95)$$

where  $\Psi_\omega^{out}(x)$  are the normalized wave functions given, from Eq. (91), by

$$\Psi_\omega^{out}(x) = e^{-i\omega v} x^{\frac{i}{\kappa_0}\omega} . \quad (96)$$

Thus, from the normalization condition

$$\langle \Psi_\omega(r) | \Psi_\omega(r) \rangle = 1 = |N_\omega|^2 \left[ e^{\frac{2\pi}{\kappa_0}\omega} - 1 \right] , \quad (97)$$

we get the resulting analogue Hawking radiation spectrum of scalar particles, which is given by

$$|N_\omega|^2 = \frac{1}{e^{\frac{2\pi}{\kappa_0}\omega} - 1} = \frac{1}{e^{\frac{\hbar\omega}{\kappa_B T_0}} - 1} . \quad (98)$$



## 4. Conclusions

In this paper, we presented analytic solutions for radial part of the Klein-Gordon equation for a massless scalar field in the both rotating and canonical acoustic black holes. These general solutions are analytic solutions for all spacetime, which means, in the region between the acoustic event horizon and infinity. The radial solution is given in terms of the confluent Heun functions, and is valid over the range  $0 \leq z < \infty$ .

The obtained results for the rotating acoustic black hole have the advantage, as compared with the one obtained in literature [47], that the solutions are valid from the exterior event horizon to infinity, instead of to be valid only close to the exterior event horizon or at infinity. Otherwise, compared with the paper by Lepe and Saavedra [48], our results are valid for any frequency, and not for a restricted range of frequency, as presented in the literature.

From these analytic solutions, we obtained the solutions for ingoing and outgoing waves near the acoustic horizon of a both rotating and canonical acoustic black holes, and used these results to discuss the Hawking radiation effect, in which we considered the properties of the confluent Heun functions to obtain the results. This approach has the advantage that it is not necessary the introduction of any coordinate system, as for example, the tortoise or Eddington-Finkelstein coordinates [54, 55, 56].

Generalizing the classical Damour-Ruffini method, we discussed the analogue Hawking radiation of both acoustic black holes. The expressions for the particle outgoing rates, given by Eqs. (47) and (94), describe the phenomena related to the radiation process for the rotating and canonical acoustic black holes, respectively.

As a final comment, we can say that we derived not only the Hawking temperature as well as the Hawking black body spectrum for both background under consideration were derived. This means that the rotating and canonical acoustic black holes behave not merely as the thermal bodies but as black bodies or dumb bodies.

## Acknowledgments

The authors would like to thank Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for partial financial support.

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